

# M. LEVIN'S CONSTRUCTION OF ABSOLUTELY NORMAL NUMBERS WITH VERY LOW DISCREPANCY

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**ABSTRACT.** Among the currently known constructions of absolutely normal numbers, the one given by Mordechay Levin in 1979 achieves the lowest discrepancy bound. In this work we analyze this construction in terms of computability and computational complexity. We show that, under basic assumptions, it yields a computable real number. The construction does not give the digits of the fractional expansion explicitly, but it gives a sequence of increasing approximations whose limit is the announced absolutely normal number. The  $n$ -th approximation has an error less than  $2^{2^{-n}}$ . To obtain the  $n$ -th approximation the construction requires, in the worst case, a number of mathematical operations that is double exponential in  $n$ . We consider variants on the construction that reduce the computational complexity at the expense of an increment in discrepancy.

## 1. INTRODUCTION

Normal numbers were introduced by Borel in 1909 [8]. A real number  $\alpha$  is normal to an integer base  $\lambda$  greater than or equal to 2 if its fractional expansion in base  $\lambda$  given by

$$\alpha - [\alpha] = \sum_{x \geq 1} \frac{d_x}{\lambda^x} \quad \text{where each } d_x \text{ is in } \{0, 1, \dots, \lambda - 1\},$$

is such that, for each positive integer  $k$ , each fixed block of digits of length  $k$  appears in  $(d_x)_{x \geq 1}$  with asymptotic frequency  $\lambda^{-k}$ . Borel calls a number absolutely normal if it is normal to every integer base greater than or equal to 2.

Let  $(\xi_x)_{x \geq 0}$  be an arbitrary sequence of real numbers in the unit interval. The quantity

$$D(P, (\xi_x)_{x \geq 0}) = \sup_{\gamma \in (0, 1]} \left| \frac{\#\{x : 0 \leq x < P \text{ and } \xi_x < \gamma\}}{P} - \gamma \right|$$

is the discrepancy of  $(\xi_x)_{x=0}^{P-1}$ . The sequence  $(\xi_x)_{x \geq 0}$  is uniformly distributed in the unit interval if  $D(P, (\xi_x)_{x \geq 0})$  goes to 0 when  $P$  goes to infinity. By a theorem of D. Wall [9, Theorem 4.14], a real number  $\alpha$  is normal to base  $\lambda$  if, and only if, the sequence  $\{\alpha \lambda^x\}_{x \geq 0}$ , where  $\{\xi\} = \xi - [\xi]$  is the fractional part of  $\xi$ , is uniformly distributed in the unit interval.

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We use the customary notation for asymptotic growth of functions, and we say  $f(n)$  is in  $O(g(n))$  if  $\exists k > 0 \exists n_0 \forall n > n_0, |f(n)| \leq k|g(n)|$ .

Borel [8] proved that almost every real number (in the sense of Lebesgue measure) is normal to every integer base and Gal and Gal [13] showed that, indeed, for almost every real number  $\alpha$  and for every integer base  $\lambda$  the discrepancy  $D(P, \{\alpha\lambda^x\}_{x \geq 0})$  is in  $O\left(\sqrt{\frac{\log \log P}{P}}\right)$ . For a thorough presentation of normal numbers and the theory of uniform distribution see the books [9, 15, 12].

In 1979 Mordechay Levin [18] considered the notion of normality for real numbers with respect to bases that are real numbers greater than 1, and he gave an explicit construction of a number that is normal to arbitrary many real bases, with controlled discrepancy of normality. More precisely, given a sequence  $(\lambda_j)_{j \geq 1}$  of real numbers greater than 1, a monotone increasing sequence  $(t_j)_{j \geq 1}$  of positive integers and a non negative real number  $a$ , Levin constructs a real number  $\alpha$  greater than  $a$  that is normal to each of the bases  $\lambda_j$ , for  $j = 1, 2, \dots$  such that  $D(P, \{\alpha\lambda^x\}_{x \geq 0})$  is in  $O\left(\frac{(\log P)^2}{\sqrt{P}}\omega(P)\right)$ , where  $\omega(P)$  is a non-decreasing function that determines from  $(\lambda_j)_{j \geq 1}$  and  $(t_j)_{j \geq 1}$  the collection of bases considered at position  $P$ , and the constant in the order symbol depends on  $\lambda_j$ . Since  $(\lambda_j)_{j \geq 1}$  and  $(t_j)_{j \geq 1}$  can be such that  $\omega(P)$  grows arbitrarily slow, so  $D(P, \{\alpha\lambda^x\}_{x \geq 0})$  can end up being in  $O\left(\frac{(\log P)^2}{\sqrt{P}}\right)$ . By considering normality with respect to arbitrary sequences  $(\lambda_j)_{j \geq 1}$  of real numbers greater than 1, Levin extends Borel's notion of absolute normality. With  $\lambda_j = j + 1$  for  $j = 1, 2, \dots$ , he obtains a number  $\alpha$  that is absolutely normal in Borel's sense.

The interest in this construction by Levin is that, among the currently known methods to construct absolutely normal numbers, it achieves the lowest discrepancy bound. In this work we give a plain presentation of this construction and we show that, under basic assumptions, the construction is computable and we establish its computational complexity.

Regarding discrepancy and computational complexity, known constructions of computable absolutely normal numbers can be classified as follows:

- Constructions that run in double exponential time, which means that to produce the  $P$ -digit of the expansion of the constructed number  $\alpha$  in a given base they perform a number of operations that is double exponential in  $P$ . One example is Alan Turing's algorithm [24, 3] for which  $D(P, \{\alpha\lambda^x\}_{x \geq 0})$  is in  $O\left(\frac{1}{\sqrt[16]{P}}\right)$ . Another is the computable reformulation of Sierpiński's construction [2] for which  $D(P, \{\alpha\lambda^x\}_{x \geq 0})$  is in  $O\left(\frac{1}{\sqrt[9]{P}}\right)$ .

- Constructions that run in exponential time, as Wolfgang Schmidt's algorithm [22] for which  $D(P, \{\alpha\lambda^x\}_{x \geq 0})$  is in  $O\left(\frac{(\log P)^4}{e^{(\log P)^{1/4}}}\right)$ . Our variants of Schmidt's algorithm [1, 6] also require exponential time. These algorithms produce numbers that are normal to all the bases in a given arbitrary set, while they are not (simply) normal to any of the multiplicatively independent bases in the complement. Besides, our algorithm [5] for computing an absolutely normal Liouville number  $\alpha$  has at least exponential complexity and we have not estimated the discrepancy of the sequence  $\{\lambda^j \alpha\}_{j=0}^{P-1}$ , for positive  $P$ .

- Constructions that run in polynomial time, as our algorithm [4] that requires just above quadratic time to compute an absolutely normal number  $\alpha$ . Speed of computation is obtained by sacrificing discrepancy. The algorithm deals explicitly with the discrepancy at the intermediate steps of the construction but we have not estimated the discrepancy of the sequence  $\{\lambda^x \alpha\}_{x \geq 0}$ .

There are constructions of numbers ensuring normality to just one base which achieve much lower discrepancy bounds than those for absolute normality. The one with smallest discrepancy was given also by Levin [19]. Using van der Corput type sequences. Levin constructs a number  $\alpha$  normal to an integer base  $\lambda$ , such that the discrepancy  $D(P, \{\alpha \lambda^x\}_{x \geq 0})$  is in  $O\left(\frac{(\log P)^2}{P}\right)$ . This discrepancy bound is surprisingly small, considering that for any sequence  $(\xi_x)_{x \geq 0}$  of reals in the unit interval,

$$\limsup_{P \rightarrow \infty} \frac{P}{\log P} D(P, (\xi_x)_{x \geq 0})$$

is greater than 0 (this result was proved by W. Schmidt in 1972, see [9]). The computational complexity of this construction has not been studied yet. Recently, Madritsch and Tichy [20] found conditions for van der Corput sets and suggest to use them for the construction of absolutely normal numbers.

The construction insuring normality to one base that has smallest computational complexity coincides with the historically first construction of a number that is normal to base 10, and it is due to Champernowne in 1933 [10]. Champernowne's constant is computable with logarithmic complexity, which means that the  $P$ -th digit in the expansion can be obtained independently of all the previous digits by performing  $O(\log P)$  elementary operations. It is also possible to compute the first  $P$  digits of Champerowne's constant in  $O(P)$  operations. The discrepancy  $D(P, \{\text{Champerowne's constant } \lambda^x\}_{x \geq 0})$  is in  $O\left(\frac{1}{\log P}\right)$  and for every  $P$  it has been proved to be greater than or equal to  $\frac{K}{\log P}$ , for positive  $K$  [10, 19, 21].

## 2. LEVIN'S CONSTRUCTION

In this section we give a comprehensible presentation of Levin's construction [18]. We reorganized the original material but we kept the notation.

**Definition.** Let  $\lambda$  be a real number greater than 1 and let  $(\lambda_j)_{j=1}^{\infty}$  a sequence of real numbers, each greater than 1. A number  $\alpha$  is normal to base  $\lambda$  if the sequence  $\{\alpha \lambda^x\}_{x \geq 0}$  is uniformly distributed in the unit interval, and absolutely normal to bases  $\lambda_j$  for all positive  $j$ , if,  $\alpha$  is normal to base  $\lambda_j$  for each positive  $j$ .

**Theorem 1** (Levin [18]). *Let  $(\lambda_j)_{j \geq 1}$  be sequence of real numbers greater than 1, let  $(t_j)_{j \geq 1}$  be a sequence of integers monotonically increasing at any speed and let  $a$  be a non-negative real number. There is a real number  $\alpha$  constructed from  $a$  and the sequences  $(\lambda_j)_{j \geq 1}$  and  $(t_j)_{j \geq 1}$  which is normal to base  $\lambda_j$  for each positive integer  $j$  and such that for any positive integer  $P$ ,*

$$D(P, \{\alpha \lambda_j^{t_j}\}_{x \geq 0}) \text{ is in } O\left(\frac{(\log P)^2 \omega(P)}{\sqrt{P}}\right)$$

where  $\omega(P) = 1$  if  $P \in [1, \ell_2)$ , and  $\omega(P) = k$  if  $P \in [\ell_k, \ell_{k+1})$ , with  $\ell_k = \max(t_k, \max_{1 \leq v \leq k} 2 \lceil \log_2 \log_2 \lambda_v \rceil + 5)$  and the constant in the order symbol depends on  $\lambda_j$ .

The number  $\alpha$  proposed by Levin is defined as

$$\alpha = a + \sum_{r=\ell_1}^{\infty} \frac{a_r}{2^{n_r} q_r},$$

$$n_r = 2^r - 2, \text{ and } q_r = 2^{2^r + r + 1}.$$

Fix  $(\lambda_j)_{j \geq 1}$  an arbitrary sequence of real numbers greater than 1, fix  $(t_j)_{j \geq 1}$  a sequence of integers monotonically increasing at any speed and fix a non-negative real  $a$ . Along the article we refer freely to the values  $\ell_r$ ,  $n_r$ ,  $q_r$ ,  $a_r$  and  $\omega(r)$  for any positive  $r$  as well as to the real  $\alpha$ .

We need some further notation. For each pair of positive integers  $r, j$  we let

$$\begin{aligned} n_{r,j} &= \lfloor n_r \log_{\lambda_j} 2 \rfloor, \\ \tau_{r,j} &= n_{r+1,j} - n_{r,j}, \text{ and} \\ A_{r,j} &= \lfloor \sqrt{\tau_{r,j}} \rfloor. \end{aligned}$$

**Lemma 2.** *For every positive  $j$  and for every  $r \geq \ell_j - 1$ ,*

$$\begin{aligned} 2^{r-1} \log_{\lambda_j} 2 &\leq \tau_{r,j} \leq 2^{r+1} \log_{\lambda_j} 2, \\ \tau_{r,j} &\geq \max(7, \tau_{r+1,j}/4). \end{aligned}$$

*Proof.* From the definitions we know that  $\tau_{r,j} = 2^r \log_{\lambda_j} 2 + \theta_{r,j}$ , where  $|\theta_{r,j}| \leq 1$ , while for  $r \geq \ell_j - 1$ ,

$$8 = 2^{\log_2 \log_2 \lambda_j + 3} \log_{\lambda_j} 2 \leq 2^r \log_{\lambda_j} 2.$$

The wanted inequalities follow.  $\square$

Fix  $\alpha_{\ell_1} = a$  and for each positive integer  $m$ , let  $a_m$  in  $[0, q_m)$ . For every  $r \geq \ell_1$  define

$$\alpha_{r+1} = \alpha_{\ell_1} + \sum_{m=\ell_1}^r \frac{a_m}{2^{n_m} q_m}.$$

We write  $e(x)$  to denote  $e^x$ . For integers  $c, m_1, m_2, r$  with  $r \geq \ell_j$  we define the quantities,

$$\begin{aligned} S_{r,j}(m_1, m_2, c) &= \sum_{x=0}^{\tau_{r,j}-1} e \left( 2\pi i \left( m_1 \left( \alpha_r + \frac{c}{2^{n_r} q_r} \right) \lambda_j^{n_{r,j}+x} + \frac{m_2 x}{\tau_{r,j}} \right) \right), \\ D_{r,j}(c) &= \sum'_{m_1, m_2 = -A_{r,j}}^{A_{r,j}} \frac{|S_{r,j}(m_1, m_2, c)|}{\overline{m_1} \overline{m_2}}, \end{aligned}$$

where  $\overline{m} = \max(1, |m|)$  and  $\sum'$  denotes that the term with  $m_1 = m_2 = 0$  is absent from the sum.

**Remark.** In Levin's paper [18] the definition of  $S_{r,j}(m_1, m_2, c)$  appears with  $\sum'$  while the definition of  $D_{r,j}(c)$  appears with  $\sum$ . However, the use of  $\sum'$  excludes the term  $m_1 = m_2 = 0$ , which only makes sense in the definition of  $D_{r,j}(c)$ .

**Construction: Levin's construction of absolutely normal numbers**

**input** : a sequence  $(\lambda_j)_{j \geq 1}$  of reals greater than 1,  
 an increasing sequence  $(t_j)_{j \geq 1}$  of integers  
 a non-negative real  $a$ .

**output:** a sequence of rationals  $(\alpha_r)_{r \geq 1}$  such that  $\lim_{r \rightarrow \infty} \alpha_r = \alpha$  and  
 for each  $\lambda_j$ , the discrepancy of  $\{\alpha \lambda_j^x\}_{x=0}^P$  is in  $O\left(\frac{(\log P)^2 \omega(P)}{\sqrt{P}}\right)$ .

**Define the function**  $\ell_k = \max(t_k, \max_{1 \leq v \leq k} 2\lceil |\log_2 \log_2 \lambda_v| \rceil + 5)$

$r = \ell_1$

$\alpha_r = a$

**repeat forever**

$n_r = 2^r - 2$

$q_r = 2^{2^r + r + 1}$

**if**  $r$  **in**  $[1, \ell_2)$  **then**  $\omega(r) = 1$

**else**  $\omega(r)$  **= the unique**  $k$  **such that**  $r$  **in**  $[\ell_k, \ell_{k+1})$

**for**  $j = 1$  **to**  $\omega(r)$  **do**

$\tau_{r,j} = n_{r+1,j} - n_{r,j}$

$A_{r,j} = \lfloor \sqrt{\tau_{r,j}} \rfloor$

**end**

**Find the least integer**  $a_r$  **in**  $[0, q_r)$  **such that for each**  $j$  **in**  $[1, \omega(r)]$

$$D_{r,j}(a_r) < 2 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2$$

**where**

$$D_{r,j}(c) = \sum'_{m_1, m_2 = -A_{r,j}}^{A_{r,j}} \frac{|S_{r,j}(m_1, m_2, c)|}{\overline{m_1} \overline{m_2}},$$

$$S_{r,j}(m_1, m_2, c) = \sum_{x=0}^{\tau_{r,j}-1} e \left( 2\pi i \left( m_1 \left( \alpha_r + \frac{c}{2^{n_r} q_r} \right) \lambda_j^{n_{r,j}+x} + \frac{m_2 x}{\tau_{r,j}} \right) \right),$$

$\sum'$  **denotes the sum without the term with**  $m_1 = m_2 = 0$ ,  
 $\overline{m} = \max(1, |m|)$ .

$$\alpha_{r+1} = \alpha_r + \frac{a_r}{2^{n_r} q_r}$$

**print**  $\alpha_{r+1}$

$r = r + 1$

**end**

**Lemma 3** (Lemma 1 in [18]). *Let integers  $j, r, m_1, m_2$  such that  $r \geq \ell_j$  and  $0 < \max(|m_1|, |m_2|) \leq A_{r,j}$ . Then,*

$$\left( \frac{1}{q_r} \sum_{c=0}^{q_r-1} |S_{r,j}(m_1, m_2, c)|^2 \right)^{1/2} < 2 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}}.$$

*Proof.* Let

$$T_{r,j}(m_1, m_2) = \left( \frac{1}{q_r} \sum_{c=0}^{q_r-1} |S_{r,j}(m_1, m_2, c)|^2 \right)^{1/2}.$$

**Remark.** Levin's original paper misses the third parameter  $c$  of the function  $S_{r,j}$ .

In accordance with the familiar inequality

$$\frac{1}{N} \left| \sum_{x=0}^{N-1} e(2\pi i \theta x) \right| \leq \min \left( 1, \frac{1}{2N \langle \langle \theta \rangle \rangle} \right),$$

where  $\langle \langle \theta \rangle \rangle$  is the distance of  $\theta$  from the nearest integer, we have

$$\begin{aligned} T_{r,j}^2(m_1, m_2) &= \\ &= \sum_{x,y=0}^{\tau_{r,j}-1} \frac{1}{q_r} \sum_{c=0}^{q_r-1} e \left( 2\pi i \left( m_1 \left( \alpha_r + \frac{c}{2^{n_r} q_r} \right) (\lambda_j^{n_{r,j}+x} - \lambda_j^{n_{r,j}+y}) + \frac{m_2(x-y)}{\tau_{r,j}} \right) \right) \\ &< \sum_{x,y=0}^{\tau_{r,j}-1} \min \left( 1, \frac{1}{2q_r \langle \langle m_1 \frac{\lambda_j^{n_{r,j}+x} - \lambda_j^{n_{r,j}+y}}{2^{n_r} q_r} \rangle \rangle} \right). \end{aligned}$$

If  $m_1$  equals 0 then  $m_2$  does not belong to  $0 \pmod{\tau_{r,j}}$ ,  $\tau_{r,j} \geq 7$ ,  $0 < |m_2| \leq A_{r,j} < \tau_{r,j}$ , and

$$T_{r,j}(0, m_2) = 0.$$

Let  $|m_1| > 0$ . Let us show that the expression under  $\langle \langle \rangle \rangle$  sign above has absolute value less than  $1/2$ . Since  $r \geq \ell_j$ , by Lemma 2,

$$\begin{aligned} \lambda_j^{n_{r+1,j}} &\leq \lambda_j^{n_{r+1} \log_{\lambda_j} 2} = 2^{n_{r+1}} = 2^{n_r} 2^{2^r}, \\ \log_{\lambda_j} 2 &= 2^{-\log_2 \log_2 \lambda_j} < 2^{\ell_j-3} < 2^{r-3}, \\ A_{r,j} &= \lfloor \sqrt{\tau_{r,j}} \rfloor < \sqrt{2^{r+1} \log_{\lambda_j} 2} < 2^{r-1}. \end{aligned}$$

Hence,

$$|m_1(\lambda_j^{n_{r,j}+x} - \lambda_j^{n_{r,j}+y})| < 2A_{r,j} \lambda_j^{n_{r+1,j}} < 2^r 2^{n_r} 2^{2^r} = (1/2) 2^{n_r} q_r,$$

and we can replace  $\langle \langle \rangle \rangle$  by the absolute value sign:

$$T_{r,j}^2(m_1, m_2) \leq \tau_{r,j} + 2 \sum_{\tau_{r,j} > x > y \geq 0} \frac{2^{n_r}}{2|m_1| \lambda_j^{n_{r,j}} (\lambda_j^x - \lambda_j^y)}.$$

Using the definition of  $n_{r,j}$ ,

$$\lambda_j^{n_{r,j}+1} \geq \lambda_j^{n_r \log_{\lambda_j} 2} = 2^{n_r},$$

whence,

$$\begin{aligned}
 T_{r,j}^2(m_1, m_2) &\leq \tau_{r,j} + \sum_{\tau_{r,j} > x > y \geq 0} \frac{1}{\lambda_j^y \lambda_j^{x-y-1} (1 - \lambda_j^{y-x})} \\
 &< \tau_{r,j} + \sum_{y,z=0}^{\infty} \frac{1}{\lambda_j^y \lambda_j^z (1 - \lambda_j^{-1})} \\
 &= \tau_{r,j} + \left( \frac{\lambda_j}{\lambda_j - 1} \right)^3 \\
 &< 4\tau_{r,j} \left( \frac{\lambda_j}{\lambda_j - 1} \right)^3.
 \end{aligned}$$

□

**Lemma 4** (Lemma 2 in [18]). *Let  $r \geq \ell_1$ . There exists an integer  $a_r$  in  $[0, q_r)$  such that, given any positive integer  $j$  and with the condition  $r \geq \ell_j$ , we have*

$$D_{r,j}(a_r) < 2 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2 \omega(r).$$

*Proof.* Using the Cauchy-Bunyakovskii-Schwarz inequality we obtain,

$$\begin{aligned}
 \frac{1}{q_r} \sum_{c=0}^{q_r-1} D_{r,j}(c) &= \sum_{m_1, m_2 = -A_{r,j}}^{A_{r,j}} \frac{1}{m_1 m_2 q_r} \sum_{c=0}^{q_r-1} |S_{r,j}(m_1, m_2, c)| \\
 &\leq \sum_{m_1, m_2 = -A_{r,j}}^{A_{r,j}} \frac{1}{m_1 m_2} \left( \frac{1}{q_r} \sum_{c=0}^{q_r-1} |S_{r,j}(m_1, m_2)|^2 \right)^{1/2}.
 \end{aligned}$$

Since the conditions of Lemma 3 are satisfied, we have

$$\begin{aligned}
 \frac{1}{q_r} \sum_{c=0}^{q_r-1} D_{r,j}(c) &< 2 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + 2 \ln A_{r,j})^2 \\
 &\leq 2 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2.
 \end{aligned}$$

Consequently, with  $r \geq \ell_j$ , the number of integers  $c$  in  $[0, q_r)$  such that

$$D_{r,j}(c) \geq 2\omega(r) \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2$$

is less than  $q_r/\omega(r)$ . By the definitions of  $\omega(r)$  and  $\ell_j$ , conditions  $r \geq \ell_j$  and  $\omega(r) \geq j$  are equivalent. In this case, the number of integers  $c$  in  $[0, q_r)$ , such that the above inequality holds for at least one positive integer  $j$ , with the condition  $r \geq \ell_j$  (alternatively,  $j \in [1, \omega(r)]$ ) is less than  $\omega(r) \lfloor q_r/\omega(r) \rfloor = q_r$ . Consequently, there exists an integer  $c = a_r$  in  $[0, q_r)$ , such that the inequality in the statement of this lemma holds for all positive  $j$  with the condition  $r \geq \ell_j$ . □

For the proof of Theorem 1 Levin uses multidimensional discrepancy and applies Koksma's inequality [14].

Let  $s$  be a positive integer, let  $\gamma_v$ , for  $v = 1, \dots, s$ , be real numbers in the unit interval, let  $(\beta_{x,v})_{x \geq 0}$  for  $v = 1, \dots, s$  be a set of real number sequences, and

let  $N_v(P)$  be the number of solutions for  $x = 0, 1, \dots, P-1$ , of the system of inequalities

$$\begin{aligned} \{\beta_{x,1}\} &< \gamma_1 \\ \{\beta_{x,2}\} &< \gamma_2 \\ &\vdots \\ \{\beta_{x,s}\} &< \gamma_s. \end{aligned}$$

The quantity

$$D(P, (\{\beta_{x,1}\}, \dots, \{\beta_{x,s}\})_{x \geq 0}) = \sup_{\gamma_1, \dots, \gamma_s \in (0,1]^s} \left| \frac{N_v(P)}{P} - \gamma_1 \cdots \gamma_s \right|$$

is called the discrepancy of the sequences  $\{\beta_{x,1}\}, \dots, \{\beta_{x,s}\}$ , for  $x = 0 \dots, P-1$ .

**Lemma 5** (Koksma [14]). *Let  $s$  be a positive integer, let  $\gamma_v$ , for  $v = 1, \dots, s$ , be real numbers in the unit interval, let  $(\beta_{x,v})_{x \geq 0}$  for  $v = 1, \dots, s$  be a set of real number sequences. Let  $P$  be a positive integer. Then, for every integer  $n$ ,*

$$D(P, (\{\beta_{x,1}\}, \dots, \{\beta_{x,s}\})_{x \geq 0}) \leq 30^s \left( \frac{1}{n} + \frac{1}{P} \sum_{m_1 \dots m_s = -n}^n \left| \frac{\sum_{x=0}^{P-1} e\left(2\pi i \sum_{v=1}^s m_v \beta_{x,v}\right)}{\overline{m}_1 \dots \overline{m}_s} \right| \right).$$

We can now present Levin's proof of Theorem 1 [18].

**Remark.** In the next proof we write  $n_{\ell_j, j}$  where Levin wrote  $n_{\ell_j}$ .

*Proof of Theorem 1.* For any three real numbers  $\xi, \lambda, \gamma$  and non-negative integers  $Q$  and  $P$ , we denote by  $N_{\xi, \lambda, \gamma}(Q, P)$  the number of solutions of the inequality

$$\{\xi \lambda^x\} < \gamma, \quad \text{for } x = Q, \dots, Q + P - 1.$$

We write  $N_{\xi, \lambda, \gamma}(P)$ , to denote  $N_{\xi, \lambda, \gamma}(0, P)$ .

Fix any positive integer  $j$  and any positive real  $\gamma$  in the unit interval. Fix any positive integer  $P$  and define an integer  $k$  from the condition  $n_{k,j} \leq P < n_{k+1,j}$ . Then,

$$P = n_{k,j} + R_1, \text{ where } 0 \leq R_1 < \tau_{k,j}.$$

Observe that when  $P$  is large enough,  $k \geq \ell_j$ . Using the definition of  $N_{\alpha, \lambda_j, \gamma}$ ,

$$N_{\alpha, \lambda_j, \gamma}(P) = N_{\alpha, \lambda_j, \gamma}(n_{\ell_j, j}) + \sum_{r=\ell_j}^k N_{\alpha, \lambda_j, \gamma}(n_{r,j}, \tau'_{r,j}),$$

where  $\tau'_{r,j} = \tau_{r,j}$  for  $r \in [\ell_j, k)$  and  $\tau'_{k,j} = R_1$ . Let us estimate  $N_{\alpha, \lambda_j, \gamma}(n_{r,j}, R)$  for  $r \geq \ell_j$  and  $0 \leq R \leq \tau_{r,j}$ . The quantity  $N_{\alpha, \lambda_j, \gamma}(n_{r,j}, R)$  is equal to the number of solutions of the system of inequalities

$$\begin{aligned} \left\{ \frac{x}{\tau_{r,j}} \right\} &< \frac{R}{\tau_{r,j}}, \\ \{\alpha \lambda_j^{n_{r,j} + x}\} &< \gamma, \end{aligned}$$



for  $x = 0, \dots, \tau_{r,j} - 1$ . We apply Lemma 5 with  $s = 2$ ,  $P = \tau_{r,j}$  and  $n = A_{r,j}$  and obtain

$$\left| N_{\alpha, \lambda_j, \gamma}(n_{r,j}, R) - \gamma \frac{R}{\tau_{r,j}} \right| \leq 30^2 \left( \frac{\tau_{r,j}}{A_{r,j}} + \sum_{m_1, m_2 = -A_{r,j}}^{A_{r,j}} \right)' \frac{1}{\overline{m_1} \overline{m_2}} \left| \sum_{x=0}^{\tau_{r,j}-1} e \left( 2\pi i \left( m_1 \alpha \lambda_j^{n_{r,j}+x} + \frac{m_2 x}{\tau_{r,j}} \right) \right) \right|.$$

Using the definition of  $\alpha_r$ , for any  $r \geq \ell_1$ ,

$$\alpha = \alpha_r + \frac{a_r}{2^{n_r} q_r} + \frac{\theta_r}{2^{n_{r+1}}},$$

where  $0 \leq \theta_r \leq 2$ , because

$$\frac{\theta_r}{2^{n_{r+1}}} = \sum_{k=r+1}^{\infty} \frac{a_k}{2^{n_k} q_k} < \sum_{k=r+1}^{\infty} \frac{1}{2^{n_k}} = \frac{1}{2^{n_{r+1}}} \sum_{k=r+1}^{\infty} \frac{1}{2^{n_k - n_{r+1}}} \leq \frac{2}{2^{n_{r+1}}}.$$

Now, using the definition of  $D_{r,j}(a_r) = \sum_{m_1, m_2 = -A_{r,j}}^{A_{r,j}}' \frac{|S_{r,j}(m_1, m_2, a_r)|}{\overline{m_1} \overline{m_2}}$  we obtain,

$$\left| N_{\alpha, \lambda_j, \gamma}(n_{r,j}, R) - \gamma R \right| \leq 30^2 \left( \frac{\tau_{r,j}}{A_{r,j}} + D_{r,j}(a_r) + \sum_{m_1, m_2 = -A_{r,j}}^{A_{r,j}}' \frac{1}{\overline{m_1} \overline{m_2}} |U(m_1, m_2, a_r)| \right)$$

where

$$|U(m_1, m_2, a_r)| = \left| S_{r,j}(m_1, m_2, a_r) - \sum_{x=0}^{\tau_{r,j}-1} e \left( 2\pi i \left( m_1 \alpha \lambda_j^{n_{r,j}+x} + \frac{m_2 x}{\tau_{r,j}} \right) \right) \right|.$$

By the definition of  $S_{r,j}(m_1, m_2, a_r)$ , the condition  $0 \leq \theta_r \leq 2$ , and the fact that for every pair of reals  $\xi_1$  and  $\xi_2$ ,

$$|e(2\pi i \xi_1) - e(2\pi i \xi_2)| = 2|\sin(\pi(\xi_1 - \xi_2))| \leq 2\pi|\xi_1 - \xi_2|,$$

we find that

$$\begin{aligned} |U(m_1, m_2, a_r)| &\leq 2\pi \sum_{x=0}^{\tau_{r,j}-1} |m_1| \lambda_j^{n_{r,j}+x} \frac{\theta_r}{2^{n_{r+1}}} \\ &\leq 4\pi |m_1| \lambda_j^{n_{r+1},j} \frac{1}{(\lambda_j - 1) 2^{n_{r+1}}} \\ &\leq \frac{4\pi |m_1|}{\lambda_j - 1}. \end{aligned}$$

Then, using that  $A_{r,j} \leq \sqrt{\tau_{r,j}}$ , the upper bound for  $D_{r,j}(a_r)$  given in Lemma 4 for  $r \geq \ell_j$ , and the inequality  $\sum_{m_1, m_2 = -A_{r,j}}^{A_{r,j}} \frac{1}{m_1 m_2} \leq (3 + \ln \tau_{r,j})^2$ , we obtain,

$$\begin{aligned} & |N_{\alpha, \lambda_j, \gamma}(n_{r,j}, R) - \gamma R| \\ & \leq 30^2 \left( 2\sqrt{\tau_{r,j}} + 2 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2 \omega(r) + \frac{4\pi}{\lambda_j - 1} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2 \right) \\ & \leq 30^2 15 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2 \omega(r). \end{aligned}$$

For  $k \geq \ell_j$ ,

$$N_{\alpha, \lambda_j, \gamma}(P) = N_{\alpha, \lambda_j, \gamma}(n_{\ell_j, j}) + \sum_{r=\ell_j}^k N_{\alpha, \lambda_j, \gamma}(n_{r,j}, \tau'_{r,j}),$$

and

$$P = n_{k,j} + R_1, \text{ where } 0 \leq R_1 < \tau_{k,j}.$$

So, we have

$$|N_{\alpha, \lambda_j, \gamma}(P) - \gamma P| \leq |N_{\alpha, \lambda_j, \gamma}(n_{\ell_j, j}) - \gamma n_{\ell_j, j}| + \sum_{r=\ell_j}^k 30^2 15 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{\tau_{r,j}} (3 + \ln \tau_{r,j})^2 \omega(r).$$

and

$$P \geq \tau_{k-1,j} \geq \frac{1}{4} \tau_{k,j}.$$

It follows from Lemma 2 that

$$\sum_{r=\ell_j}^k \sqrt{\tau_{r,j}} \leq \sum_{r=\ell_j}^k \sqrt{2^{r+1} \log_{\lambda_j} 2} \leq 3 \sqrt{2^{k+2} \log_{\lambda_j} 2} \leq 10 \sqrt{\tau_{k,j}}.$$

Let us show that, for  $k \geq \ell_j$ ,

$$\omega(P) \geq \omega(k).$$

Since  $\omega(r)$  is a non-decreasing sequence, it is sufficient to show that, for  $k \geq \ell_j$ ,

$$P \geq k.$$

In fact, using the definitions of  $\ell_j$  and  $n_{k,j}$ , and  $P = n_{k,j} + R_1$ ,

$$k \geq \ell_j \geq 5, \quad 2^{\frac{k+1}{2}} \geq k+1 \text{ for } k \geq 5,$$

and

$$\begin{aligned} P - k & \geq n_{k,j} - k \\ & \geq (2^k - 2) \log_{\lambda_j} 2 - k - 1 \\ & \geq (\log_{\lambda_j} 2) (2^{k-1} - (k+1) \log_2 \lambda_j) \\ & \geq (\log_{\lambda_j} 2) (2^{k-1} - (k+1) 2^{\frac{\ell_j-3}{2}}) \\ & \geq 2^{\frac{k-3}{2}} (\log_{\lambda_j} 2) (2^{\frac{k+1}{2}} - k - 1) \\ & \geq 0. \end{aligned}$$

Using the inequalities above and the obvious inequality  $|N_{\alpha, \lambda_j \gamma}(n_{\ell_j, j}) - \gamma n_{\ell_j, j}| \leq n_{\ell_j, j}$ , we have

$$|N_{\alpha, \lambda_j \gamma}(P) - \gamma P| \leq n_{\ell_j, j} + 4 \cdot 10^5 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{P} (5 + \ln P)^2 \omega(P).$$

The above inequality also holds for  $k \leq \ell_j - 1$ , since

$$|N_{\alpha, \lambda_j \gamma}(P) - \gamma P| \leq P < n_{k+1, j} \leq n_{\ell_j, j}.$$

Recalling the definition of  $n_{\ell_j, j}$  we finally obtain

$$|N_{\alpha, \lambda_j \gamma}(P) - \gamma P| \leq 2^{\ell_j} \log_{\lambda_j} 2 + 4 \cdot 10^5 \left( \frac{\lambda_j}{\lambda_j - 1} \right)^{3/2} \sqrt{P} (5 + \ln P)^2 \omega(P).$$

Thus, the discrepancy of the sequence  $\{\alpha \lambda_j^x\}_{x \geq 0}$ , for any given positive integer  $P$ ,

$$D(P, \{\alpha \lambda_j^x\}_{x \geq 0}) = \sup_{\gamma \in (0, 1]} \left| \frac{N_{\alpha, \lambda_j \gamma}(P)}{P} - \gamma \right| \text{ is in } O\left(\frac{(\log P)^2}{\sqrt{P}} \omega(P)\right).$$

□

**Corollary 6** ([18]). *Let  $\lambda_j = j + 1$ ,  $t_j = 2^j$  for  $j = 1, 2, \dots$ , so  $\ell_j \leq 2^{j+1} + 1$  and  $\omega(P) \leq 2(5 + \ln P)$ . Then, the constructed number  $\alpha$  is absolutely normal in Borel's sense, and for any integer  $j \geq 2$ , the discrepancy of  $\{\alpha j^x\}$ , for  $x = 0, \dots, P - 1$  is*

$$D(P, \{\alpha j^x\}_{x \geq 0}) \leq \frac{2^{2j+1}}{P} \log_j 2 + 3 \cdot 10^6 \frac{(5 + \ln P)^3}{\sqrt{P}},$$

which is in  $O\left(\frac{(\log P)^3}{\sqrt{P}}\right)$ .

Levin asserts that a similar method can be used for constructing a number  $\alpha$  such that, given any integer  $j$ , the discrepancy of the sequence  $\{\alpha \lambda_j^x\}_{x=0}^{P-1}$ , is  $O\left(\frac{(\log P)^{3/2}}{\sqrt{P}} \omega(P)\right)$ , where the constant in the order symbol  $O$  depends on  $\lambda_j$ , and he gives as reference Section 2 of [17].

### 3. ABOUT LEVIN'S CONSTRUCTION AND ITS POSSIBLE VARIANTS

**3.1. Possible variants on the construction.** Here we consider other possible values for  $n_r$  and  $q_r$  to run Levin's construction. This is interesting because smaller values of  $n_r$  imply a faster computation at step  $r$ , due to the fact that  $a_r$  is searched in a smaller range. However, smaller values of  $n_r$  imply a larger discrepancy of the sequence  $\{\lambda_j^x \alpha\}_{x \geq 0}$ .

The next Lemma 7 gives a sufficient condition for  $n_r$  and  $q_r$  to ensure that the construction works. Then, Lemma 8 gives a sufficient condition on  $n_r$  to ensure that the construction yields an absolutely normal number: the value  $n_r$  must be polynomial in  $r$ , with degree greater than 1.

**Lemma 7.** *If  $\lambda_j \geq 2$  and the sequences  $n_1, n_2, \dots$  and  $q_1, q_2, \dots$  satisfy for every positive  $r$ ,*

$$2^{n_{r+1} - n_r + 1 + \frac{1}{2} \log(n_{r+1} - n_r + 1)} \leq q_r$$

*then the statement of Lemma 3 holds.*

*Proof.* In Lemma 3, every step of the proof is valid disregarding the values chosen for  $n_1, n_2, \dots$  and  $q_1, q_2, \dots$  except for the statement

$$|m_1|(\lambda_j^{n_{r,j}+x} - \lambda_j^{n_{r,j}+y}) \leq \frac{1}{2} 2^{n_r} q_r.$$

We show that the condition given by this lemma is sufficient to make the above inequality true. Let us recall that  $n_{r,j} = \lfloor n_r \log_{\lambda_j} 2 \rfloor$ ,  $\tau_{r,j} = n_{r+1,j} - n_{r,j}$ ,  $0 \leq x, y < \tau_{r,j}$  and  $|m_1| \leq A_{r,j} = \lfloor \sqrt{\tau_{r,j}} \rfloor$ . Then,

$$\begin{aligned} q_r &\geq 2^{n_{r+1}-n_r+1+\frac{1}{2}\log_2(n_{r+1}-n_r+1)} \\ &= \sqrt{n_{r+1}-n_r+1} 2^{n_{r+1}-n_r+1} \\ &\geq \sqrt{(n_{r+1}\log_{\lambda_j} 2 - n_r \log_{\lambda_j} 2) + 1} 2^{n_{r+1}-n_r+1} \\ &\geq \sqrt{n_{r+1,j} - n_{r,j}} 2^{n_{r+1}-n_r+1} \\ &= \sqrt{\tau_{r,j}} 2^{n_{r+1}-n_r+1} \\ &\geq |m_1| 2^{n_{r+1}-n_r+1} \\ &= 2|m_1| 2^{n_{r+1}} 2^{-n_r} \\ &> 2|m_1| \lambda_j^{n_{r+1,j}} \lambda_j^{-(n_{r,j}+1)} \\ &= 2|m_1| \lambda_j^{n_{r+1,j}-n_{r,j}-1} \\ &= 2|m_1| \lambda_j^{\tau_{r,j}-1} \\ &> 2|m_1| (\lambda_j^{\tau_{r,j}-1} - 1) \\ &\geq 2|m_1| \frac{\lambda_j^{n_{r,j}}}{2^{n_r}} (\lambda_j^{\tau_{r,j}-1} - 1) \\ &\geq 2|m_1| \frac{\lambda_j^{n_{r,j}}}{2^{n_r}} (\lambda_j^x - \lambda_j^y) \\ &= \frac{2}{2^{n_{r,j}}} |m_1| (\lambda_j^{n_{r,j}+x} - \lambda_j^{n_{r,j}+y}). \end{aligned}$$

□

In what follows we use customary asymptotic notation to describe the growth rate of the functions. We write,

$$\begin{aligned} f(n) \text{ is in } \Theta(g(n)) &\text{ if } \exists k_1 > 0 \exists k_2 > 0 \exists n_0 \forall n > n_0, \quad k_1 g(n) \leq f(n) \leq k_2 g(n), \\ f(n) \text{ is in } o(g(n)) &\text{ if } \forall k > 0 \exists n_0 \forall n > n_0, \quad |f(n)| \leq k|g(n)|. \end{aligned}$$

**Lemma 8.** Let  $j$  and  $P$  be positive integers and let  $k$  be such that  $n_{k,j} \leq P < n_{k+1,j}$ .

If  $\sum_{r=1}^k \sqrt{n_{r+1,j} - n_{r,j}}$  is in  $o\left(\frac{P}{(\log P)^2 \omega(P)}\right)$  then Levin's construction yields an absolutely normal number.

*Proof.* See proof of Theorem 1 for the upper bound of  $|N_{\alpha, \lambda_j, \gamma}(P) - \gamma P|$ . □

The next proposition shows that if  $n_r$  dominates any linear function on  $r$ , and  $q_r$  is increasing in  $r$  according to a condition in the the growth of  $n_r$ , then Levin's construction yields an absolutely normal number,

**Proposition 9.** Let  $(\lambda_j)_{j \geq 1}$  be a sequence of real numbers greater than 1 and let  $(t_j)_{j \geq 1}$  be a sequence of reals such that the function  $\omega(P)$  has sub-polynomial

growth. If  $n_r$  is any polynomial on  $r$  with degree greater than 1, and  $q_r$  is such that

$$n_{r+1} - n_r + 1 + \frac{1}{2} \log(n_{r+1} - n_r + 1) \leq \log q_r,$$

then Levin's construction yields an absolutely normal number. However, if  $n_r$  is linear in  $r$ , Levin's arguments do not prove that the discrepancy goes to 0.

*Proof.* Suppose  $n_r$  is polynomial on  $r$ . Then, there is some  $h$  such that  $n_r$  is in  $\Theta(r^h)$ . By definition of  $n_{r,j}$ , we have  $n_{r,j} = \lfloor n_r \log_{\lambda_j} 2 \rfloor$  is in  $\Theta(r^h)$ . Hence,  $n_{r+1,j} - n_{r,j}$  is in  $\Theta(r^{h-1})$ ; therefore,  $\sqrt{n_{r+1,j} - n_{r,j}}$  is in  $\Theta(r^{\frac{h-1}{2}})$ . Furthermore, if  $P$  and  $k$  are such that  $n_{k,j} \leq P < n_{k+1,j}$ , then  $k$  is in  $\Theta(\sqrt[h]{P})$ . Thus,

$$\sum_{r=1}^k \sqrt{n_{r+1,j} - n_{r,j}} \text{ is in } \Theta\left(\left(\sqrt[h]{P}\right)^{\frac{h+1}{2}}\right) = \Theta(P^{\frac{h+1}{2h}}).$$

If  $n_r$  were a linear function on  $r$ ,

$$\sum_{r=1}^k \sqrt{n_{r+1,j} - n_{r,j}} \text{ is in } \Theta(P),$$

hence  $\sum_{r=1}^k \sqrt{n_{r+1,j} - n_{r,j}}$  would not be in the required class  $o\left(\frac{P}{(\log P)^2 \omega(P)}\right)$ . We conclude that, to obtain a normal number with Levin's construction,  $n_r$  can not be linear in  $r$ . Instead,  $n_r$  can be any polynomial on  $r$  with degree greater than 1 provided that  $\omega(P)$  is chosen to have sub-polynomial growth.  $\square$

In Levin's construction smaller values of  $n_r$  imply a larger upper bound on discrepancy of the sequence  $\{\alpha \lambda^x\}$ . The following table shows the bound for the discrepancy of the sequence  $\{\lambda_j^x \alpha\}_{x=0}^P$ , obtained using Levin's proof for different choices of  $n_r$ . In each case the constant behind the  $O$  symbol depends on  $\lambda_j$ .

$n_r$	Discrepancy bound given by Levin's proof
$r$	$O(\log(P)^2 \omega(P))$ —it does not go to 0 when $P$ goes to $\infty$ —
$r^h$	$O\left(\frac{\log(P)^2 \omega(P)}{P^{\frac{h-1}{2h}}}\right)$
$2^r - 2$	$O\left(\frac{\log(P)^2 \omega(P)}{\sqrt{P}}\right)$

In all these cases, the upper bound for discrepancy contains  $\omega(P)$ , as in Levin's formulation and the constant hidden in the  $O$  symbol depends on the base  $\lambda_j$ . Although Levin states that for any nondecreasing function  $\omega(P)$  his construction produces an absolutely normal real number, the growth of  $\omega(P)$  cannot be arbitrary. For example, when  $n_r$  is  $2^r - 2$ ,  $\omega(P) = \sqrt{P}$  does not give a discrepancy bound going to 0.

**3.2. Necessary conditions on the construction.** Levin's construction is not conceived as the concatenation of the binary expansions of the  $a_r$  for  $r = 1, 2, \dots$ . This means that the expansion in base 2 of  $\alpha_{r+1}$  is *not* obtained as a concatenation of the expansion of  $\alpha_r$  with the base-2 representation of  $a_r$ . Recall the definition of  $\alpha_{r+1}$ :  $\alpha_{\ell_1}$  is equal to a starting real number  $a$  (argument for the construction) and for every  $r \geq \ell_1$ ,

$$\alpha_{r+1} = \alpha_{\ell_1} + \sum_{m=\ell_1}^r \frac{a_m}{2^{n_m} q_m},$$

where  $a_m$  is an integer in  $[0, q_m)$  satisfying the conditions of Lemma 4,

$$n_m = 2^m - 2 \text{ and } q_m = 2^{2^m + m + 1}.$$

Since  $\log q_r = 2^r + r + 1 > n_{r+1} - n_r = 2^r$  we have

$$\alpha_{r+1} - \lfloor 2^{n_{r+1}} \alpha_{r+1} \rfloor 2^{-n_{r+1}} > 0.$$

The next Lemma 10 shows that if  $q_r$  is unbounded, then it is necessary for Levin's proof that  $q_r > 2^{n_{r+1} - n_r}$ . This condition is implied by the sufficient condition on  $n_r$  and  $q_r$  we identified in Lemma 7.

Then, Lemma 11 proves that if  $q_r$  is bounded then Levin's construction does not yield an absolutely normal number.

**Lemma 10.** *If  $q_r$  is unbounded then it is necessary that  $q_r > 2^{n_{r+1} - n_r}$ .*

*Proof.* For Lemma 3 to hold, we need that  $|m_1|(\lambda_j^{n_{r,j}+x} - \lambda_j^{n_{r,j}+y}) \leq \frac{1}{2} 2^{n_r} q_r$ . In particular, when  $m_1 = A_{r,j}$ ,  $x = \tau_{r,j} - 1$ ,  $y = 0$  and  $\lambda_j = 2$ , we need that the following inequality holds:

$$A_{r,j}(2^{n_{r+1}-1} - 2^{n_r}) \leq \frac{1}{2} 2^{n_r} q_r.$$

Equivalently,

$$A_{r,j}(2^{n_{r+1}-n_r} - 2) \leq q_r.$$

Now suppose that,  $q_r$  is unbounded, non-decreasing in  $r$  but, contrary to the statement of the Lemma,  $q_r \leq 2^{n_{r+1}-n_r}$ . So, the above condition becomes

$$A_{r,j} \leq \frac{q_r}{(q_r - 2)}.$$

Since  $q_r$  is unbounded, there is  $r_0$  such that for every  $r \geq r_0$ ,  $q_r \geq 1000$ , and each of the following inequalities should hold.

$$\begin{aligned} A_{r,j} &\leq \frac{q_r}{q_r - 2} \leq \frac{1000}{998} < 2 \\ A_{r,j} &= \lfloor \sqrt{n_{r+1} - n_r} \rfloor < 2 \\ \sqrt{n_{r+1} - n_r} &< 3 \\ n_{r+1} - n_r &< 9 \\ 2^{n_{r+1}-n_r} &< 512. \end{aligned}$$

Then, using the assumption  $q_r \leq 2^{n_{r+1}-n_r}$ , we conclude  $q_r < 512$ , contradicting that  $q_r \geq 1000$ .  $\square$

The following lemma shows that if  $q_r$  is bounded by a constant, then Levin's construction yields a number  $\alpha$  which might not be absolutely normal.

**Lemma 11.** *If  $q_r$  is bounded by a constant and  $\log q_r \leq n_{r+1} - n_r$  then Levin's construction does not ensure absolute normality.*

*Proof.* For ease of presentation assume the argument  $a$  in Levin's construction is a non-negative rational number of the form  $r + p/2^{\ell_1}$  for some non-negative integers  $r$  and  $p$  with  $p$  less than  $\ell_1$ . So, the expansion of  $a$  in base 2 has at most  $2^{\ell_1}$  significant digits. The case where  $a$  is not of this form can be proved similarly.

Suppose that  $q_r$  is bounded, then  $\log_2 q_r$  will be bounded too. That is, there is a constant  $C$  such that for all  $r$ ,  $\log_2 q_r \leq C$ . Since  $a_r$  is in  $[0, q_r)$ , at step  $r$ , the choice of  $a_r$  requires at most  $C$  binary digits. Suppose also that  $\log_2 q_r \leq n_{r+1} - n_r$ , which implies that the fractions  $\frac{a_r}{2^{n_r q_r}}$  have binary expansions that not overlap. Therefore, the first  $n_r$  bits of  $\alpha$  will be correctly computed on the  $r$ -th step of the construction, that is the first  $n_r$  bits of  $\alpha$  will coincide with those of  $\alpha_r$ .

Let  $0.b_0b_1b_2\dots$  be the binary expansion of  $\alpha$  and let  $\text{count}(\alpha, n, b)$  be the number of bits equal to  $b$  within  $b_0, b_1, \dots, b_{n-1}$ . Assuming  $\alpha$  is absolutely normal, it must be simply normal in base 2. Therefore  $\lim_{n \rightarrow \infty} \frac{\text{count}(\alpha, n, b)}{n} = \frac{1}{2}$  for  $b$  in  $\{0, 1\}$ . Using the definition of limit, for all positive  $\epsilon$ , for every sufficiently large  $r$ ,

$$\frac{1}{2} - \epsilon < \frac{\text{count}(\alpha, n_r, 1)}{n_r} \leq \frac{C(r-1)}{n_r}.$$

So,

$$n_r < \frac{Cr - C}{\frac{1}{2} - \epsilon}.$$

Since this holds all positive  $\epsilon$ , we conclude

$$n_r \text{ in } O(r).$$

On the other hand, we can safely assume that  $\log_2 q_r \geq 1$  because at least 1 bit should be computed on each step of the algorithm. Given that  $n_{r+1} - n_r \geq \log_2 q_r \geq 1$  we obtain that  $n_r$  must be in  $\Theta(r)$ . As we stated on Proposition 9, a linear growth of  $n_r$  does not ensure that discrepancy goes to 0. So we cannot ensure absolute normality of the generated number  $\alpha$ .  $\square$

More importantly, this necessary condition on  $n_r$  and  $q_r$  determines that Levin's construction of the number  $\alpha$  is not doable as a concatenation of the  $a_r$ , for  $r = 1, 2, \dots$

**Proposition 12.** *If  $q_r$  and  $n_r$  are such that  $\log q_r > n_{r+1} - n_r$  then Levin's construction of  $\alpha$  is not doable as the concatenation of the  $a_r$ , for  $r = 1, 2, 3, \dots$*

*Proof.* To run the construction as a concatenation of the  $a_r$ , for  $r = 1, 2, 3, \dots$ , we need that

$$\sum_{m=0}^{r-1} \log q_m \leq n_r.$$

But

$$\sum_{m=0}^{r-1} \log q_m > \sum_{m=0}^{r-1} n_{m+1} - n_m = n_r - n_0 = n_r.$$

□

#### 4. LEVIN'S NORMAL NUMBERS ARE COMPUTABLE

The theory of computability defines a computable function from non-negative integers to non-negative integers as one which can be effectively calculated by some algorithm. The definition extends to functions from one countable set to another, by fixing enumerations of those sets. A real number  $x$  is computable if there is a base and a computable function that gives the digit at each position of the expansion of  $x$  in that base. Equivalently, a real number is computable if there is a computable sequence of rational numbers  $(r_n)_{n \geq 0}$  such that  $|x - r_n| < 2^{-n}$  for each  $n \geq 0$ .

**Theorem 13** (Turing [11, Theorem 5.1.2]). *The following are equivalent:*

- (1) *The real  $x$  is computable.*
- (2) *There is a computable sequence of rationals  $(r_n)_{n \geq 0}$  that tends to  $x$  such that  $|x - r_n| < 2^{-n}$  for all  $n$ .*
- (3) *There is a computable sequence of rationals  $(r_n)_{n \geq 0}$  that converges to  $x$  and a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $|x - r_{f(n)}| < 2^{-n}$  for all  $n$ .*

**Theorem 14.** *Let  $(\lambda_j)_{j \geq 1}$  be computable sequence of integers greater than 2 let  $(t_j)_{j \geq 1}$  be a computable sequence of integers monotonically increasing at any speed and let the starting value  $a$  be a rational number, Then, the number  $\alpha$  defined by Levin, proved to be absolutely normal in Theorem 1, is computable.*

*Proof.* The number  $\alpha$  is the limit of  $\alpha_r$  for  $r$  going to infinity, where  $a_{\ell_1} = a$  with  $\ell_1 = \max(t_1, 2\lceil \log_2 \log_2 \lambda_1 \rceil + 5)$ , and for  $r \geq 1$ ,

$$\alpha_{r+1} = \alpha_r + \frac{a_r}{2^{n_r} q_r},$$

where  $a_r$  is an integer in  $[0, q_r)$  satisfying the inequalities of Lemma 4,  $n_r = 2^r - 2$  and  $q_r = 2^{2^r + r + 1}$ . Lemma 4 proves that such  $a_r$  exists. Since  $D_{r,j}(c)$  is a computable function it is possible to find  $a_r$  by an exhaustive search among all integers in  $[0, q_r)$  and all bases  $\lambda_j$  for  $j = 1, 2, \dots, \omega(r)$ , where  $\omega(r) = 1$  if  $r$  in  $[1, \ell_2)$ , otherwise  $\omega(r)$  is the unique index  $k$  such that  $r$  in  $[\ell_k, \ell_{k+1})$ , with  $\ell_k = \max(t_k, \max_{1 \leq v \leq k} 2\lceil \log_2 \log_2 \lambda_v \rceil + 5)$ . At each step  $r$ , we can compute bitwise approximations of  $D_{r,j}$  from above, for each of the possible candidate values of  $a_r$  until we find one that satisfies the requires inequality for all  $j$  between 1 and  $\omega(r)$ . Thus, the sequence of rationals  $\alpha_1, \alpha_2, \dots$  is computable and converges to an absolutely normal number  $\alpha$ . From the proof of Theorem 1 we know that, for each  $r$ ,

$$|\alpha - \alpha_r| < \frac{2}{2^{n_r}}.$$

Since  $\alpha$  is an absolutely normal number, and therefore an irrational number, by Theorem 13 we conclude that  $\alpha$  is computable. □

#### 5. THE COMPUTATIONAL COMPLEXITY OF LEVIN'S CONSTRUCTION

Theorem 14 proves that under some assumptions of the sequences  $(\lambda_j)_{j \geq 1}$  and  $(t_j)_{j \geq 1}$ , and the starting value  $a$ , Levin's construction is indeed an algorithm to compute the number  $\alpha$ . The algorithm is recursive.



The standard computational model is the Turing machine model, which works just with finite representations, so it only deals with numbers that are the limit of a computable sequence of finite approximations. In this model, at step  $r$ , the number of elementary operations needed to find out the number  $a_r$  can not be easily determined. This is because to find out  $a_r$  the algorithm must compute sums of exponential sums. The terms in these sums are transcendental numbers, which can only be computed as limits of finite approximations. It is impossible to determine how many approximations to each term of the exponential sums must be computed to find out that a candidate  $a_r$  is conclusive.

So, instead of counting the number of elementary operations needed to compute the number  $a_r$  at step  $r$ , here we give the number of mathematical operations needed in an idealized computational model over the real numbers, based on machines with infinite-precision real numbers. A canonical model for this form of computation over the reals is Blum-Shub-Smale machine [7], abbreviated BSS machine. This is a machine with registers that can store arbitrary real numbers and can compute rational functions over reals at unit cost. Since elementary transcendental functions, as exponential function or trigonometric functions, are not computable by a BSS machine we need to consider the extended BSS machine which includes exponential and trigonometric functions as primitive operations. For our purpose, the extended BSS model is identical to considering Boolean arithmetic circuits augmented with trigonometric functions.

Of course, for any given real valued function, its complexity in the BSS model gives just a lower bound of its complexity in the classical Turing machine model, where the cost for arithmetic (and trigonometric) operations over the real numbers is not constant.

**Theorem 15.** *Let  $(\lambda_j)_{j \geq 1}$  be a computable sequence of reals greater than 1 and let  $(t_j)_{j \geq 1}$  be a computable sequence of integers. Levin's algorithm requires*

$$O\left(2^{2^r+3r+1} \sum_{j=1}^{\omega(r)} (\log_{\lambda_j} 2)^2\right)$$

*mathematical operations to compute  $\alpha_r$ , for each  $r$ .*

*Proof.* Assume a BSS machine which includes exponential and trigonometric functions as primitive operations. The expression  $S_{r,j}(m_1, m_2, c)$  is the sum of  $\tau_{r,j}$  terms, each of them can be computed in constant time in our machine. Hence the time needed to compute each value of  $S_{r,j}$  is in  $O(\tau_{r,j})$ .

To obtain a value of  $D_{r,j}$  we must calculate  $O(A_{r,j}^2) = O(\tau_{r,j}^2)$  values of  $S_{r,j}$ . Therefore, the computation of  $D_{r,j}$  is in  $O(\tau_{r,j}^2) = O((2^r \log_{\lambda_j} 2)^2)$ .

Finding the value of  $a_r$  requires to compute  $D_{r,j}(c)$  for each  $j$  between 1 and  $\omega(r)$  until we find a value of  $c$  in  $[0, q_r]$  which satisfies the inequalities of Lemma 4. In the worst case, it will be necessary to try all possible values for  $c$ . In this worst case, the required time is in

$$O\left(\sum_{c=0}^{q_r-1} \sum_{j=1}^{\omega(r)} (2^r \log_{\lambda_j} 2)^2\right) = O\left(q_r \sum_{j=1}^{\omega(r)} (2^r \log_{\lambda_j} 2)^2\right) = O\left(2^{2^r+3r+1} \sum_{j=1}^{\omega(r)} (\log_{\lambda_j} 2)^2\right).$$

Let  $C_k$  be the time required to compute  $a_k$ ,

$$C_k = 2^{2^k+3k+1} \sum_{j=1}^{\omega(k)} (\log_{\lambda_j} 2)^2$$

Then, the time to compute  $\alpha_r$  is  $\sum_{k=1}^r C_k$ . We now show that the time needed to compute  $\alpha_r$  is essentially the time spent on the search of  $a_r$ . We need to show that  $\sum_{k=1}^r C_k$  is in  $O(C_r)$ , because

$$\sum_{k=1}^{r-1} C_k \leq (r-1)C_{r-1} = (r-1)2^{2^{r-1}+3(r-1)+1} \sum_{j=1}^{\omega(r-1)} (\log_{\lambda_j} 2)^2$$

which is in  $O\left(2^{2^r+3r+1} \sum_{j=1}^{\omega(r)} (\log_{\lambda_j} 2)^2\right)$ .  $\square$

Notice that Theorem 15 estimates the complexity of obtaining a rational approximation  $\alpha_r$  with an error bounded by  $2^{-(n_{r+1}-1)}$ . Since  $\alpha_r$  is just an approximation to  $\alpha$ , it is not determined how many bits in the expansion of  $\alpha_r$  are conclusive so as to conform the expansion of  $\alpha$ . One would like that the first  $n_{r+1} - 1$  bits of  $\alpha_r$  determine those of  $\alpha$ . As we showed in Proposition 12 Levin's construction is not doable as the concatenation of the values  $a_r$ . An overlapping of the fractions  $\frac{a_r}{2^{n_r} q_r}$  may occur, causing carries and changing some of the first bits of  $\alpha_r$ .

Theorem 15 proves that the complexity of computing  $\alpha_r$  with Levin's original formulation for  $n_r$  and  $q_r$ , is double exponential in  $r$ . Since  $n_r$  is the number of bits of  $\alpha_r$  that are obtained at step  $r$ , and in Levin's original formulation  $n_r$  is  $2^r - 2$ , it is fair to say that the complexity of Levin's algorithm is simply exponential in the number of bits computed at step  $r$ .

We now prove that, in case  $n_r$  is quadratic in  $r$ , then Levin's algorithm requires a number of operations that is simply exponential in the square root of number of bits computed at step  $r$ .

**Theorem 16.** *The alternative of Levin's construction with  $n_r = r^2$  takes*

$$O\left(r^3 2^{2r} \sum_{j=1}^{\omega(r)} (\log_{\lambda_j} 2)^2\right)$$

*mathematical operations in an extended BSS machine to compute  $\alpha_r$ .*

*Proof.* First, we need to choose values for  $q_r$  that ensure normality. As we showed in Lemma 7,  $2^{n_{r+1}-n_r+1+\frac{1}{2}\log(n_{r+1}-n_r+1)} \leq q_r$  is a sufficient condition. We choose  $q_r = 2^{2r+2+\lceil \log(2r+2) \rceil}$ . By Theorem 15, to find  $a_r$ , in the worst case it is necessary to compute  $D_{r,j}(c)$  for each  $j$  between 1 and  $\omega(r)$  and for each  $c$  between 0 and  $q_r - 1$  and each  $D_{r,j}$  requires  $O(\tau_{r,j}^2)$  operations. Then, the number of operations

to find  $a_r$  is in  $O(q_r \sum_{j=1}^{\omega(r)} \tau_{r,j}^2)$ , because

$$q_r \text{ is in } O(r2^{2r}),$$

$$\tau_{r,j} \text{ is in } O(r \log_{\lambda_j} 2), \text{ and}$$

$$O\left(q_r \sum_{j=1}^{\omega(r)} \tau_{r,j}^2\right) = O\left(r^3 2^{2r} \sum_{j=1}^{\omega(r)} (\log_{\lambda_j} 2)^2\right).$$

The time to compute  $\alpha_r$  is essentially the time required to find  $a_r$  because

$$\sum_{k=1}^r k^3 2^{2k} \sum_{j=1}^{\omega(k)} (\log_{\lambda_j} 2)^2 \leq r^3 2^{2r+2} \sum_{j=1}^{\omega(r)} (\log_{\lambda_j} 2)^2$$

which is in  $O\left(r^3 2^{2r} \sum_{j=1}^{\omega(r)} (\log_{\lambda_j} 2)^2\right)$ .  $\square$

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